Sets, Induction, Counting

1 Sets and Relations

Question 1:

Using the basic definition of subsets, show that the empty set is a subset of every set.

key:

1. Subset definition: $A \subseteq B : \forall x \in A \rightarrow x \in B$.
2. Empty set definition: an empty set has no element.
   Let $A = \emptyset$. we need to prove that $\forall x \in \emptyset \rightarrow x \in B$.
3. Prove by controdiction: suppose that $\exists x \in \emptyset \rightarrow x \in B$
4. Vacuously true: you cannot pick any $x \in \emptyset$.
   Thus, no such x exists.

$\Rightarrow$ The empty set is a subset of every set.

Question 2:

Definitions: A relation R on any set A is said to be:

Reflexive: if $\forall x \in A, \quad (x, x) \in R$.

Transitive: if $\forall x, y, z \in A, \quad ((x, y) \in R \ AND \ (y, z) \in R) \Rightarrow (x, z) \in R$.

Symmetric: if $\forall x, y \in A, \quad (x, y) \in R \Rightarrow (y, x) \in R$.

Asymmetric: if $\forall x, y \in A, \quad (x, y) \in R \Rightarrow (y, x) \notin R$.

For each of the following relations, state which of the above five properties hold. We will represent the relation using the symbol R.

(a) For $x, y \in \mathbb{Z}$, $(x, y) \in R \Rightarrow |x| = |y|$.

(b) For $x, y \in \mathbb{Z}$, $(x, y) \in R \Rightarrow x < y$.

(c) For $x, y \in \mathbb{Z}$, $(x, y) \in R \Rightarrow x + y$ is even.

key:

(a) Reflexive, Symmetric, Transitive.

(b) Transitive, Asymmetric.

(c) Reflexive, Symmetric, Transitive.

Any reasonable explain is accepted.

Note: Reflexive is true for $\leq, =$, and $|$, false for $>, <, \neq, \nmid$.

2 Induction

Question 1:

Prove the following proposition

$$P(n): \quad 2n + 1 < 2^n \quad \text{for all } n \geq 3$$

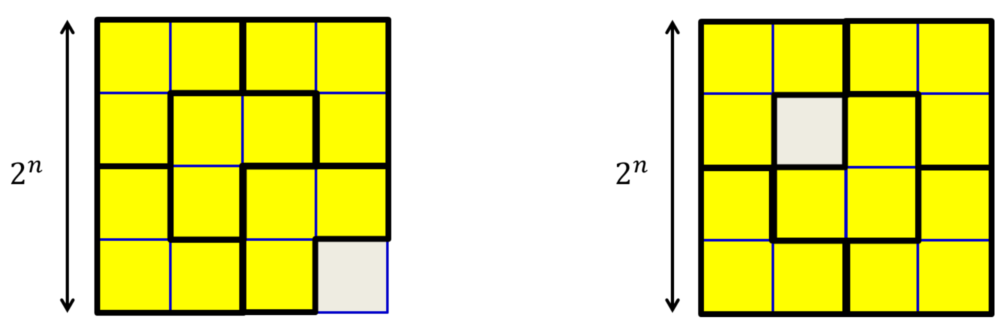
key:

1. Base case: n = 3, 7 < 8 is true.
2. Assume it is true for $n = k$, that is
$$2k + 1 < 2^k$$
3. Let's consider the case when $n = k + 1$,
$$2(k + 1) + 1 = (2k + 1) + 2 < 2^k + 2^k = 2^{k+1}$$
Thus, the proposition holds true for $n = k + 1$.

$\Rightarrow$ We can prove by induction that $P(n): \quad 2n + 1 < 2^n \quad$ for all $n \geq 3$ is true.

Question 2:

We have seen in class how a chess board of size $2^n, n \geq 2$ can be completely filled using L-shaped blocks, leaving one square empty as shown in the following figures. In class we used the specific construction that the empty square is at a corner of the board. Using induction, prove that this empty square could, in fact, be placed *anywhere* in the board.

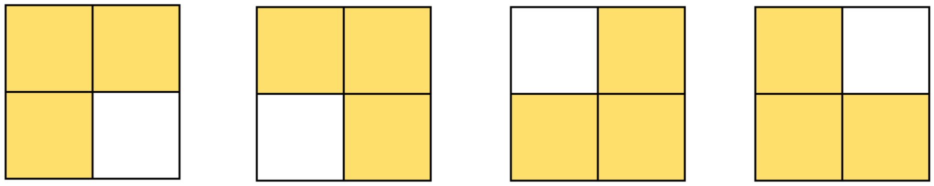


Left: A board of size 2^n ($n = 2$) filled using L-shaped blocks of size 3, leaving a corner square empty.

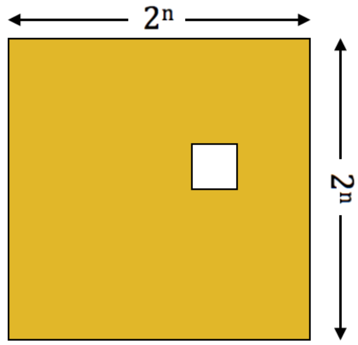
Right: Filling the same board leaving a different square empty.

key:

1. Base case: n = 2, obviously it is true.

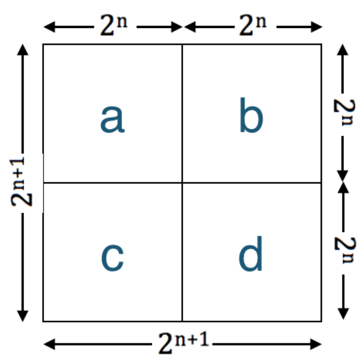


2. Assume it is true for $n = k$, that is, the empty square could be placed anywhere in the board with size of 2^k.

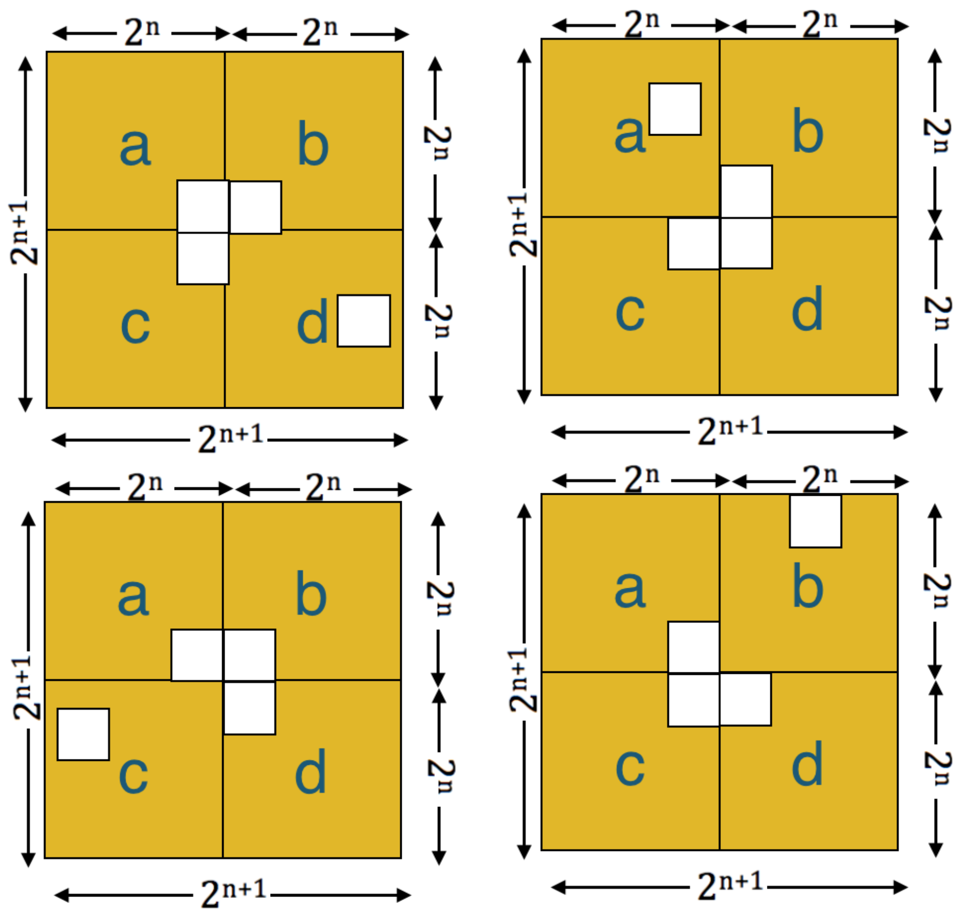


3. Let's consider when $n = k + 1$,

$(2^{k+1})^2 = 4 * (2^k)^2$, which means a square of size $2^n + 1$ can be decomposed to 4 squares of size 2^n.



We know from the assumption that every smaller board of size 2^n can be placed an empty square anywhere. Let any 3 of these 4 empty squares form a L-shaped block at the centre of the big square, and the last empty square can be placed in its board anywhere.



Question 3:

Using induction, prove that every day is the same day of the week. Formally, you must prove the hypothesis P(n) that All days in any set of n days are the same day of the week. In order to do so, you must make a questionable assumption of the base case. Explain the problem. Wikipedia will give you the wrong answer, so don't bother with it.

key:

1. If you assume $n = 1$ is the base case, you can prove "all days in any set of n days are the same day of the week" is true.

2. But by common sense, we know that "all days in any set of n days are the same day of the week" is false. For example, Monday is different with Tuesday. Where is the problem? Is the induction a wrong method in some cases?

3. Let's clasify the difference between "equailvlent" and "equal":

   equailvlent is a relationship, which is built based on two different objects.

   equal means the object itself.

4. Actually $n = 2$ is the right base case, which cannot be hold by the proposition.

$\Rightarrow$ Days in one week are not the same.

3 Counting

Question 1:

Prove that the set $\{1, 2, 7, 9, 10, 12\}$ is finite.

key:

This set has an isomorphism with J_6.

Question 2:

Show that the intersection of a finite set and a countable set is finite.

key:

Let $\mathbb{F}$ be the finite set and $\mathbb{C}$ be the countable set, and $\mathbb{S} = \mathbb{C} \cap \mathbb{F}$

1. $\mathbb{F}$ has an isomorphism with J_f, for some $f \in \mathbb{N}$.

2. An element in $\mathbb{S}$ if and only if it is in both $\mathbb{C}$ and $\mathbb{F}$.

3. $\mathbb{S}$ contains s of elements, $s \leq f$. Thus, $\mathbb{S}$ have an isomorphism with J_s, for some $s \in \mathbb{N}$. That is, $\mathbb{S}$ is finite.

$\Rightarrow$ The intersection of a finite set and a countable set is finite.

Question 3:

Show that the union of two finite sets is finite.

key: Let $\mathbb{A}$ and $\mathbb{B}$ be the two finite sets, and $\mathbb{S} = \mathbb{A} \cup \mathbb{B}$.

1. A has an isomorphism with J_a, for some $a \in \mathbb{N}$. B has an isomorphism with J_b, for some $b \in \mathbb{N}$.

2. An element in $\mathbb{S}$ if it is in any one of $\mathbb{C}$ or $\mathbb{F}$.

3. $\mathbb{S}$ has at most $a + b$ of elements, say s, $s \leq a + b$.

4. Thus, $\mathbb{S}$ can have an isomorphism with J_s, for some $s \in \mathbb{N}$, that is, $\mathbb{S}$ is finite.

$\Rightarrow$ The union of two finite sets is finite.

4 Axioms

Question 1:

A *field* is a set for which we have defined an "addition" operation, usually denoted by "+", and a "multiplication" operation, usually denoted by "×", which satisfy the following axioms. We will denote the field by the symbol F below.

Axioms of addition: A1: F is closed under addition. If $x \in F$ and $y \in F$, then $x + y \in F$.

A2: Addition is commutative. $x + y = y + x$.

A3: Addition is associative. $x + (y + z) = (x + y) + z$.

A4: F contains an element 0, the "additive identity element", such that for all $x \in F$, $x + 0 = x$.

A5: For every $x \in F$, there is a corresponding "additive inverse" $y \in F$ such that $x + y = 0$. We will represent this additive inverse as "$-x$".

Axioms of multiplication: M1: F is closed under multiplication. If $x \in F$ and $y \in F$, then $x \times y \in F$.

M2: Multiplication is commutative. $x \times y = y \times x$.

M3: Multiplication is associative. $x \times (y \times z) = (x \times y) \times z$.

M4: F contains a "multiplicative identity" element $1 \neq 0$, such that for all $x \in F$, $x \times 1 = x$.

M5: For every $x \in F$, if $x \neq 0$, there is a corresponding "multiplicative inverse" $y \in F$ such that $x \times y = 1$. We will represent this multiplicative inverse as "$\frac{1}{x}$".

The distributive law: for all $x, y, z \in F$,

$$x \times (y + z) = x \times y + x \times z.$$

Note that the above are axioms. They cannot be derived. They are simply established by definition or diktat.

Fun Facts: *You may be surprised to realize that commutativity, associativity, distributivity, and the non-equivalence of additive and multiplicative identities are not demonstrable facts. These are rules established by axiom. We can eliminate one or more of these rules, and a different type of mathematics will result, as we will see later.*

Using the axioms of multipication, prove the following.

(a) If $x \neq 0$ and $x \times y = x \times z$, then $y = z$.

(b) If $x \neq 0$ and $x \times y = x$, then $y = 1$.

(c) If $x \neq 0$ and $x \times y = 1$, then $y = \frac{1}{x}$.

(d) If $x \neq 0$, then $\frac{1}{\frac{1}{x}} = x$.

key:

(a)

1. $x \neq 0 \rightarrow \frac{1}{x}$ $\quad M5$

2. $y = 1 \times y$ $\quad M4$

$= (\frac{1}{x} \times x) \times y$ $\quad M5$

$= \frac{1}{x} \times (x \times y)$ $\quad M3$

$= \frac{1}{x} \times (x \times z)$

$= (\frac{1}{x} \times x) \times z$ $\quad M5$

$= 1 * z$ $\quad M5$

$= z$ $\quad M4$

(b)

1. $x \neq 0 \rightarrow \frac{1}{x}$ $\quad M5$

2. $y = 1 \times y$ $\quad M4$
$= (\frac{1}{x} \times x) \times y$ $\quad M5$
$= \frac{1}{x} \times (x \times y)$ $\quad M3$
$= \frac{1}{x} \times x$
$= 1$ $\quad M5$

(c)

1. $x \neq 0 \rightarrow \frac{1}{x}$ $\quad M5$

2. $y = 1 \times y$ $\quad M4$
$= (\frac{1}{x} \times x) \times y$ $\quad M5$
$= \frac{1}{x} \times (x \times y)$ $\quad M3$
$= \frac{1}{x} \times 1$
$= \frac{1}{x}$ $\quad M4$

(d)

1. $x \neq 0 \rightarrow \frac{1}{x} \neq 0$ $\quad M5$

2. $\frac{1}{\frac{1}{x}} = \frac{1}{\frac{1}{x}} \times 1$
$= \frac{1}{\frac{1}{x}} \times (\frac{1}{\frac{1}{x}} \times x)$
$= 1 \times x$
$= x$

Note: we cannot use $A = B \rightarrow \frac{1}{x} \times A = \frac{1}{x} \times B$ directly, as it is not the axiom.