Reals, Complex Numbers, Counting

1 The Reals

Question 1:

Are $\sqrt{3}$ and $\sqrt{5}$ rational numbers? And how about $\sqrt{5}-\sqrt{3}$? Justify your answer with proofs.

key:

(a) $\sqrt{3}$ and $\sqrt{5}$ are not rational numbers.

1. Suppose $\sqrt{3}$ is rational number,

we can get $\sqrt{3}=\frac{a}{b}$, where a and b have no common factors besides 1.

2. a^2 is a multiple of 3 as $a^2=3b^2$, so a should be a multiple of 3 and let a = 3k.

3. We can get that b must also be a multiple of 3 as $b^2=3k^2$.

Thus, a and b have common factor 3, which is a contradiction.

$\Rightarrow \sqrt{3}$ is not a rational number.

This is the same applies to $\sqrt{5}$ by (a).

$\Rightarrow \sqrt{3}$ and $\sqrt{5}$ are not rational numbers.

(b) $\sqrt{5}-\sqrt{3}$ is not rational numbers.

1. Suppose $r=\sqrt{5}-\sqrt{3}$ is rational,

2. $r^2=(\sqrt{5}-\sqrt{3})^2=8-2\sqrt{15}$

$\rightarrow 2\sqrt{15}=8-r^2$.

3. $8-r^2$ is rational number because the set of rational number is closed under multiplication and addition.

4. We can prove that $\sqrt{15}$ is not rational number by (a),

5. thus, $2\sqrt{15}$ is irrational, which is a contradiction.

$\Rightarrow \sqrt{5}-\sqrt{3}$ is not a rational number.

Question 2:

Prove that

(a) between any two real numbers, there is a rational one.

(b) between any two real numbers, there are uncountable number of irrationals.

key:

(a) Refer to Rudin Chapter 1 (1.20).

(b) Let a, b be two real numbers, $a \leq b$,

$\mathbb{S}$: a set of real numbers in (a,b),

$\mathbb{S}_r$: a set of rational number in $\mathbb{S}$.

$\mathbb{S}_{ir}$: a set of irrational number in $\mathbb{S}$.

1. $\mathbb{S}$ is uncountable by cantors diagonalization.

2. $\mathbb{S}_r$ is countable, since $\mathbb{S} \in \mathbb{Q}$ and $\mathbb{Q}$ is countable.

3. Suppose that $\mathbb{S}_{ir}$ is countable, then we get $\mathbb{S}$ is countable as $\mathbb{S} = \mathbb{S}_r \cup \mathbb{S}_{ir}$, which is a contradiction.

$\Rightarrow$ There are uncountable number of irrational numbers between any two real numbers.

2 The Complex Field

Question 1:

Let a and b be complex numbers (a,b $\in \mathbb{C}$), prove that

(a) $|a| \geq 0$,

(b) $|\overline{a}| = |a|$,

(c) $|ab| = |a||b|$,

(d) $|\operatorname{Re}(a)| \leq |a|$,

(e) $|a + b| \leq |a| + |b|$.

key:

Refer to Rudin Chapter 1 (1.33).

Question 2:

Let $Z_1, Z_2, Z_3, \ldots, Z_n \in \mathbb{C}$, prove that

(a)

$$|Z_1 + Z_2 + Z_3 + \cdots + Z_n| \leq |Z_1| + |Z_2| + |Z_3| + \cdots + |Z_n|$$

(b)

$$|< Z_1, Z_2 > + < Z_2, Z_3 > + \cdots + < Z_{n-1}, Z_n > + < Z_n, Z_1 > | \leq |Z_1|^2 + |Z_2|^2 + \cdots + |Z_n|^2$$

key:

(a)

1. Let's prove that $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$.

Let $Z_1 = x_1 + y_1 i$, and $Z_2 = x_2 + y_2 i$, in which x_1, x_2, y_1, y_2 are all reals.

$$|Z_1 + Z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$
$$|Z_1| + |Z_2| = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

We need to prove that $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$.

Square both sides,

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \leq x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$x_1 x_2 + y_1 y_2 \leq \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

Square both sides, we can get

$$2x_1 x_2 y_1 y_2 \leq x_1^2 y_2^2 + y_1^2 x_2^2,$$

$$0 \leq (y_1 x_2 - x_1 y_2)^2$$

This is true.

$\Rightarrow |Z_1 + Z_2| \leq |Z_1| + |Z_2|$

2. Since $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$,

$|Z_2 + Z_3| \leq |Z_2| + |Z_3|$,

$\ldots,$

$|Z_{n-1} + Z_n| \leq |Z_{n-1}| + |Z_n|$,

We can get,

$|Z_1 + Z_2| + \cdots + |Z_{n-1} + Z_n|$

$\leq |Z_1 + Z_2 + \cdots + Z_{n-1} + Z_n|$

$\leq |Z_1| + |Z_2| + |Z_3| + \cdots + |Z_n|$

(b)

$$\begin{aligned}
&| < Z_1, Z_2 > + < Z_2, Z_3 > + \cdots + < Z_{n-1}, Z_n > + < Z_n, Z_1 > | \\
&= ||Z_1||Z_2| \cos\theta_1 + |Z_2||Z_3| \cos\theta_2 + \cdots + |Z_{n-1}||Z_n| \cos\theta_{n-1} + |Z_n||Z_1| \cos\theta_n \\
&\leq ||Z_1||Z_2| + |Z_2||Z_3| + \cdots + |Z_{n-1}||Z_n| + |Z_n||Z_1|| \\
&\leq \frac{1}{2} \times |2|Z_1||Z_2| + 2|Z_2||Z_3| + \cdots + |2Z_{n-1}||Z_n| + 2|Z_n||Z_1|| \\
&\leq \frac{1}{2} \times |(|Z_1|^2 + |Z_2|^2) + (|Z_2|^2 + |Z_2|^2) + \cdots + (|Z_{n-1}|^2 + |Z_n|^2) + (|Z_n|^2 + |Z_1|^2)| \\
&\leq \frac{1}{2}|2|Z_1|^2 + 2|Z_2|^2 + \cdots + 2|Z_n|^2| \\
&\leq |Z_1|^2 + |Z_2|^2 + \cdots + |Z_n|^2
\end{aligned}$$

3 Counting

Question 1:

Is the set of all irrational real numbers countable? Prove your answer.

key:

1. $\mathbb{R}$ is uncountable.
2. $\mathbb{Q}$ is countable.
3. Let $ir\mathbb{Q}$ be the set of irrational numbers.
   Suppose that $ir\mathbb{Q}$ is countable,
   so that $\mathbb{R}$ is countable as $\mathbb{R} = \mathbb{Q} \cup ir\mathbb{Q}$, which is a contradition.

$\Rightarrow$ The set of irrational real numbers uncountable.

Question 2:

Prove that every infinite subset of a countable set is countable.

key:

Refer to Rudin Chapter 2 (2.8).

Question 3:

Show that any finite set of reals includes its infimum. You must use induction for this proof.

key:

1. Base case: when a set contains only one element: $\mathbb{A} = \{a\}$.
   a is the infimum and included in $\mathbb{A}$, which apparently holds.
2. Assume it applys to any finite set of size k, say, $\mathbb{B} = \{b_1, b_2, \ldots, b_k\}$,
   and b_i is the infimum, $b_i \in \mathbb{B}$.
3. Consider the finite set with size of $k + 1$, say $\mathbb{C} = \{c_1, c_2, \ldots, c_k, c_{k+1}\}$.
   $\mathbb{C}$ has infimum, as it is a finite set.
   Let $\mathbb{C}' = \mathbb{C} \setminus \{c\}, c \in \mathbb{C}$,
   Thus, $\mathbb{C}$ of size k includes its infimum c' by assumption.
   1. If $c' \geq c$, c is the infimum of $\mathbb{C}$.
   2. If $c' < c$, c' is the infimum of $\mathbb{C}$.

   Therefore $\mathbb{C}$ includes its infimum.

$\Rightarrow$ Any finite set of reals includes its infimum.

Question 4:

Show that any finite set of reals includes its supremum.

key: This applies the same as question 3.