Metric Spaces and Sets

1 DeMorgan's Laws

Question 1:

Let $\{E_\alpha\}$ be a collection of sets in any metric space. The subscript α is used to indicate that the collection may not be countable.

Prove the following (The superscript c represents a complement relative to the metric space):

(i)

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$

Hint: You may find it easier to prove that if any element x belongs to the LHS implies that it belongs to the RHS and vice versa.

(ii)

$$\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$$

key:

(i) Consider $x \in (\bigcup_\alpha E_\alpha)^c$

$\Rightarrow x \notin \bigcup_\alpha E_\alpha$

$\Rightarrow x \notin$ any $E_\alpha, \forall \alpha$

$\Rightarrow x \notin E_{\alpha_1} \& E_{\alpha_2} \& E_{\alpha_3} \dots$

$\Rightarrow x \in E_{\alpha_1}^c \& x \in E_{\alpha_2}^c \& x \in E_{\alpha_3}^c \dots$

$\Rightarrow x \in E_{\alpha_1}^c \cap E_{\alpha_2}^c \cap E_{\alpha_3}^c \dots$

$\Rightarrow x \in \bigcap_\alpha E_\alpha^c$

$\Rightarrow (\bigcup_\alpha E_\alpha)^c \subseteq \bigcap_\alpha E_\alpha^c$

Consider $x \in \bigcap_\alpha E_\alpha^c$

$\Rightarrow x \in E_{\alpha_1}^c \& x \in E_{\alpha_2}^c \& x \in E_{\alpha_3}^c \dots$

$\Rightarrow x \notin E_{\alpha_1} \& x \notin E_{\alpha_2} \& x \notin E_{\alpha_3} \dots$

$\Rightarrow x \notin E_{\alpha_1} \cup E_{\alpha_2} \cup E_{\alpha_3} \dots$

$\Rightarrow x \notin \bigcup_\alpha E_\alpha$

$\Rightarrow x \in (\bigcup_\alpha E_\alpha)^c$

$\Rightarrow \bigcup_\alpha E_\alpha^c \subseteq (\bigcap_\alpha E_\alpha)^c$

Thus, we can arrive at the conclusion that

$$\left(\bigcap_{\alpha} E_\alpha\right)^c = \bigcup_{\alpha} E_\alpha^c$$

(ii) This applies the same as (i).

Question 2:

Prove the following:

(i) The arbitrary union of open sets is open.

Hint: Recall that an open set, by definition, is one in which every element is an interior point.

(ii) The arbitrary intersection of closed sets is closes.

Hint: De Morgan's laws.

(iii) Finite intersection of open sets is open.

(iv) Finite union of closed sets is closed.

key:

(i) Given an arbitrary collection of open sets, say $\{G_\alpha\}$, we need to prove that $\bigcup_\alpha G_\alpha$ is open.

Consider $x \in \bigcup_\alpha G_\alpha$

$\Rightarrow x \in G_{\alpha_1} \cup G_{\alpha_2} \cup G_{\alpha_3} \dots$

$\Rightarrow x \in G_{\alpha_1}$ or $x \in G_{\alpha_2}$ or $x \in G_{\alpha_3} \dots$

$\Rightarrow x$ is interior point for some G_α.

$\Rightarrow \exists$ a neighborhood of $x, N_r(x) \in G_\alpha$ for some α.

$\Rightarrow N_r(x) \in \bigcup_\alpha G_\alpha$

$\Rightarrow x$ is interior point of $\bigcup_\alpha G_\alpha$

$\Rightarrow \bigcup_\alpha G_\alpha$ is an open set.

Thus, we can arrive at the conclusion that the arbitrary union of open sets is open.

(ii) Given an arbitrary collection of closed sets, say $\{G_\alpha\}$, we need to prove that $\bigcap_\alpha G_\alpha$ is closed.

If G_α is closed, then ${G_\alpha}^c$ is open.

$\Rightarrow \bigcup_\alpha G_\alpha^c$ is open (the arbitrary union of open sets is open).

$\Rightarrow (\bigcap_\alpha G_\alpha)^c$ is open $((\bigcap_\alpha G_\alpha)^c = \bigcup_\alpha G_\alpha^c)$.

$\Rightarrow \bigcap_\alpha G_\alpha$ is closed.

Thus, we can arrive at the conclusion that the arbitrary intersection of closed sets is closed.

(iii) Given a finite collection of open set, say $\{G_i\}$, we need to prove that $\bigcap_{i=0}^n G_i$ is open.

Consider $x \in \bigcap_{i=0}^{n} G_i$

$\Rightarrow x \in G_1$ and $x \in G_2$ and $x \in G_3 \ldots x \in G_n$

$\Rightarrow x$ is interior point for all G_i.

$\Rightarrow \exists$ neighborhoods of $x_i, N_r(x_i) \in G_i$ with radius of r_i, can be covered by G_i individually.

$\Rightarrow N_r(x_{min}) \in$ all G_i with the minimum radius ($r_{min} = min\{r_1, r_2, \ldots, r_n\}$).

(The minimim must exits as $\{r_i\}$ is finite.)

$\Rightarrow N_r(x_{min}) \in \bigcap_{i=0}^{n} G_i$

$\Rightarrow x_{min}$ is an interior point in $\bigcap_{i=0}^{n} G_i$.

$\Rightarrow \bigcap_{i=0}^{n} G_i$ is open.

Thus, we can arrive at the conclusion that finite intersection of open sets is open.

(iv) This applies the same as (ii) by using (iii) and $(\bigcup_{i=0}^{n} G_i)^c = \bigcap_{i=0}^{n} G_i^c$.

2 Open and Closed Sets

Question 1:

Show that

(i) The (relative) complement of an open set is closed.

(ii) The (relative) complement of closed set is open.

key:

(i) Consider an open set $\mathbb{E}$ in metric space $(\mathbb{X}, d)$.

$\mathbb{E}^c = \mathbb{X} \setminus \mathbb{E} = \{p \in \mathbb{X} : p \notin \mathbb{E}\}$.

Since $\mathbb{E}$ is open, all of the points in $\mathbb{E}$ are interior points.

$\Rightarrow$ for all $x \in \mathbb{E}$, $\exists$ a neighborhood N_r of $x, N_r(x) \subseteq \mathbb{E}$.

$\Rightarrow N_r(x)$ is disjoint from $\mathbb{E}^c$ for all x.

$\Rightarrow \forall x \in \mathbb{E}, x$ is not a limit point of $\mathbb{E}^c$.

$\Rightarrow \mathbb{E}^c$ has no limit point outside itself.

$\Rightarrow \mathbb{E}^c$ contains all its limit points.

$\Rightarrow \mathbb{E}^c$ is closed.

We can arrive at the conclusion that the (relative) complement of an open set is closed.

(ii) Consider a closed set $\mathbb{E}$ in metric space $(\mathbb{X}, d)$, and $\mathbb{E}^c$ is the complement of $\mathbb{E}$ in $\mathbb{X}$.

$\Rightarrow \mathbb{E}$ has all its limit points.

$\Rightarrow$ Consider $x \in \mathbb{E}^c$, x is not a limit point of $\mathbb{E}$ as $x \notin \mathbb{E}$.

$\Rightarrow \exists$ a neighborhood N_r of $x, N_r(x) \cap \mathbb{E} = \emptyset$.

$\Rightarrow N_r(x) \subseteq \mathbb{E}^c$.

$\Rightarrow x$ is an interior point of $\mathbb{E}^c$.

$\Rightarrow \mathbb{E}^c$ is an open set.

We can arrive at the conclusion that the (relative) complement of closed set is open.

Question 2:

Consider any set C. Let C' be the set of limit points of C. The closure of C is defined as $\bar{C} = C \cup C'$. Show that $\bar{C}$ is also closed.

key:

Consider p is a limit point of $\bar{C}$, we need to prove that $p \in \bar{C}$.

Since $\bar{C} = C \cup C'$, how to prove that $p \in C$ or $p \in C'$?

1. If $p \in C$, we find such point p.
2. If $p \notin C$, we need to prove that $p \in C'$, that is, p is a limit point of C.

   Consider a neighbohood $N_r(p)$ of p, we need to find $N_r(p)$ contains a point that is also in C.

   Since p is a limit point of $\bar{C}$, $N_r(p)$ contains a point, say, $q \in \bar{C}$.

   1. If $q \in C$, we find such point q.
   2. If $q \notin C$, then q is a limit point of C ($\bar{C} = C \cup C'$).

      Consider a neighbohood $N_r(q)$ of q, such that $N_r(q) \subset N_r(p)$.

      $\exists$ a point $q' \in N_r(q)$, so $q' \in N_r(p)$, then we find such point q'.

$\Rightarrow$ Thus, we can arrive at the conclusion that $\bar{C}$ is closed.

3 Compact Sets

Show the following. In general, for proofs relating to compactness, we draw upon the fact than any cover has a finite subcover, and we now only have deal with a finite number of elements. Recall also that finite collections of numbers provably have a supremum and an infimum within the collection.

(i) Finite sets are compact.

key: Consider the elements in a finite set $\mathbb{S}$ are: $x_1, x_2, \ldots, x_N$, and an open cover set $\{G_\alpha\}$ covering $\mathbb{S}$.

For $\forall x_i$, choose one G_{α_i} that contains x_i.

$\Rightarrow \{G_\alpha\}_{i=0}^k$ covers $x_1, x_2, \ldots, x_N, k \leq N$ (since one of G_{α_i} may contains more than one element in $\mathbb{S}$).

$\Rightarrow \{G_\alpha\}_{i=0}^k$ is a finite subcover.

$\Rightarrow$ and hence finite sets are compact.

(ii) Compact sets are bounded.

key:

Consider K is a compact set $\rightarrow$ Any open cover of K has a finite subcover.

Consider the open cover $\{B(x_i) : x_i \in K\}_{i=0}^{N}$, each $B(x_i)$ has a radius of 1.

$\{B(x_i)\}$ has a finite cover $\rightarrow \exists$ a subcover $\{B(x_i)\}_{i=0}^{k}, k \leq N$.

Since $x_1, x_2, \ldots, x_N, k \leq N$ is finite, $R = max_{0<i,j<N}\{d(x_i, x_j)\}$ must exists.

Thus, $\exists$ a point $p \notin K$, $B(p)$ with the radius of (R + 2) covers all K.

There, K is bounded set.

$\Rightarrow$ Thus, compact sets are bounded.

(iii) Compact sets are closed.

key:

Consider a compact set K, we need to prove K^c is open.

For $p \in K^c$, and $r_x = \frac{1}{2}d(x,p)$ for each $x \in K$.

Consider a collection of neighborhoods $\{N_{r_x}(x) : x \in K\}$, it is an open cover of K, and so has a finite subcover.

$\exists$ a neighborhood of x, $N_r(p)$, such that $N_{r_x}(x) \cap N_r(p) = \emptyset$ (Please draw the pictures, then it will be obvious).

$\Rightarrow N_r(p) \cap (N_{r_{x_1}} \cup N_{r_{x_2}} \cup \ldots N_{r_{x_n}}) = \emptyset$

$\Rightarrow N_r(p) \notin K$

$\Rightarrow N_r(p) \in K^c$

$\Rightarrow p$ is an interior point of K^c.

$\Rightarrow K^c$ is an open set.

$\Rightarrow K$ is closed.

Thus, compact sets are closed.

(iv) For any set C (in a metric space)

$$C \text{ compact} \Longleftrightarrow C \text{ closed and bounded}$$

Note that you have to prove this both ways, as this is a bidirectional relation.

key:

I. C compact $\Rightarrow C$ closed and bounded has been proved by (ii) and (iii).

II. Prove that C closed and bounded $\Rightarrow C$ compact.

Actually this is not true for arbitrart metric space, for example, $\mathbb{X} = \{\frac{1}{n} : n \in N^*\}$ in discrete.

Here I will prove it in R^n.

First, let's introduce a concept: k-cell.

k-cell is a set in the form of $[a_1, b_1] \times \cdots \times [a_k, b_k]$ in $\mathbb{R}^k$, where $a_i < b_i$ for $i = 1, \ldots, k$.

We can think of k-cell as a k-dimensional rectangular region.

Second, le's prove that every k-cell is compact.

Suppose k-cell is not compact.

Consider $\{G\}$ is an open cover of k-cell, thus $\{G\}$ has no finite subcover.

We will split the k-cell in $\mathbb{R}^n$ in half in every dimension.

In 1-dimension, split $[-r, r]$ into $[-r, 0]$ and $[0, r]$.

In 2-dimensions, split the rectangle into four equivalent rectangles. And so on ....

Thus a k-cell is divided into 2^k subcells, and each of them is also a k-cell.

Since subcell $\subseteq$ cell, some subcollection of $\{G\}$ must be an open cover of each subcell. Thus at least one of the subcells have an infinite cover. For these subcells with an infinite cover, subdivide them by cutting it in half in each dimension again. Then we can construct an infinite sequence of subcells and some sets in this infinite sequence are from $\{G\}$.

Since every set in the subcollection is an open set, thus these contains the point from $\{G\}$ also contains the subcell, so the subcell has a finite open cover, which is contraditory.

We can arrive at that the k-cell is compact.

Finally we can prove closed and bounded set is compact by using k-cell.

If C is bounded, we have $C \subset$ a collection I of some k-cells. I is compact since every k-cell is compact. Since C is closed subset of a compact set, C is compact.

(v) If C is closed and K is compact, $C \cap K$ is compact.

key:

We know that $C \cap K \subset K$ and K is compact, thus K is closed and bounded, and $C \cap K$ is also bounded (the same ball which bounds K can also bounds $C \cap K$).

Since K and C are closed, thus $C \cap K$ is closed.

Thus, $C \cap K$ is bounded and closed $\Rightarrow$ compact.