1 Probability Axioms

Question 1:

Let A,B,C be three arbitrary events. Find the probability of exactly one of these events occuring.

solution:

Sample space S: $\{ABC, AB, AC, BC, A, B, C, \emptyset\}$, and $|S| = 8$.

P(A or B or C) $= \frac{3}{8}$.

note: other reasonable answers are accepted.

Question 2:

If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, show that $P(B) < P(B|A)$

solution: Given $P(A) > 0$, $P(B) > 0$,

$$P(A) < P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A) \cdot P(B) < P(A \cap B)$$

$$P(B) < \frac{P(A \cap B)}{P(A)} = P(B|A)$$

2 Simple Counting

Question 1:

Cards are dealt, one at a time, from a standard 52-card deck.

1. If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades?
2. If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades?
3. If the first 4 cards are all spades, what is the probability that the next card is also a spade?

solution:

1. $$P = \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = \frac{33}{3920}$$
2. $$P = \frac{10}{49} \cdot \frac{9}{48} = \frac{15}{392}$$

3.
$$P = \frac{9}{48}$$

Question 2:

Five cards are drawn from a standard 52-card playing deck. What is the probability that all 5 cards will be of the same suit?

solution:

There are $\binom{52}{5} = 2598960$ combinations of drawing five cards from a full deck, and $\binom{13}{5} = 1287$ of drawing 5 cards in one of 4 suits.

Therefore,
$$P = \frac{4 \times 1287}{2598960} = \frac{33}{16660}$$

Question 3:

If two balanced dices are rolled, what the is the probability that the difference between the two numbers that appear will be less than 3?

solution:

There are $6 \times 6 = 36$ possible results of rolling two balanced dices, and 24 combinations of two numbers with the difference less than 3.

Therefore,
$$P = \frac{24}{36} = \frac{2}{3}$$

3 Independence

Question 1:

Suppose that A and B are mutually exclusive events, with $P(A) > 0$ and $P(B) < 1$. Are A and B independent? Prove your answer.

solution:

$P(A|B) = 0$ since A and B are mutually exclusive.

$P(A \cap B) = P(A|B) \times P(B) = 0$

Suppose A and B are independent, $P(A \cap B) = P(A) \cdot P(B) = 0$.

Since $P(A) > 0$ and $P(B) < 1$, $P(B) = 0$.

Consider $P(B|A) = P(B) = 0$, and $P(A|B)$ is undefined with $P(B) = 0$.

Thus, A and B are independent.

Question 2:

Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Are A and B independent? Prove your answer.

solution:

Since $A \subset B$, $P(B|A) = 1$.

$$P(A \cap B) = P(B|A)P(A) = P(A)$$

Suppose A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$, we have $P(B) = 1$.

Consider $P(A|B) = P(A)$ and $P(B|A) = P(B) = 1$, we have A and B are independent only if $P(B) = 1$.

Question 3:

Suppose that a balanced coin is independently tossed two times. Define the following events:

A: Head appears on the first toss.

B: Head appears on the second toss

C: Both tosses yield the same outcome

Are A, B, and C mutually independent?

solution:

$P(A) = P(B) = \frac{1}{2}$ and $P(C) = \frac{2}{2\times 2} = \frac{1}{2}$.

$P(A \cap B \cap C) = \frac{1}{2\times 2} \neq P(A) \cdot P(B) \cdot P(C)$.

Thus, A, B, and C are not mutually independent.

4 Combinatorics, Bayes

Question 1:

If $2n$ teams are to be assigned to games $1, 2, \ldots, n$, in how many ways can the teams be assigned to the games?

solution:

Here we assume that a game consists of 2 different team and A plays with B is same as B plays with A.

First make an order of all teams and assign every two without duplicate to one team in order. There are $(2n)!$ combinations. For every game, we don't need to care the order of 2 teams, thus, divide the number of permutations by 2 for each team.

Thus, there are $\frac{(2n)!}{2^n}$ combinations.

Question 2:

There is a box with two fair (equal probability of each side) coins in it. One coin has heads on both sides, the other one has heads and tails like a regular coin. You pick a coin randomly from the box and without looking at it, spin it and observe heads on the top. What is the probability that the other side is also heads?

solution:

Consider three events,

A_1: Pick the coin with heads on both sides.

A_2: Pick the coin that has head and tail.

H: Head apprears.

$P(A_1) = P(A_2) = \frac{1}{2}$

$P(H|A_1) = \frac{1}{2}, P(H|A_2) = 1$

$P(A_1|H) = \frac{P(A_1 \cap H)}{P(H)} = \frac{P(A_1)P(H|A_1)}{P(A_1)P(H|A_1)+P(A_2)P(H|A_2)} = \frac{2}{3}$

Question 3:

If n letters are placed at random in n envelopes, what is the probability that exactly $n-1$ letters will be placed in the correct envelopes?

solution:

Since it is impossible to have exactly $n-1$ letters in the correct envelopes and remaind 1 in the wrong place, thus, its pobability is 0.

Question 4:

If A and B are mutually exclusive events and $P(B) > 0$, show that $P(A|A \cup B) = \frac{P(A)}{P(A)+P(B)}$.

solution:

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$$

Since A and B are mutually exclusive,

$$A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup \emptyset = A$$

$$P(A \cup B) = P(A) + P(B)$$

Thus,

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

5 Conditional Probability

Question 1:

We have two dice (of 6 faces). Let the outcomes of the dice be X,Y and let A = X+Y, B = max(X,Y). Find the conditional probability of A, given that B is even.

solution:

There are 21 possible combinations for B being even.

$$P(A=1|B \text{ is even}) = 0, P(A=2|B \text{ is even}) = 0, P(A=3|B \text{ is even}) = \frac{2}{21}$$
$$P(A=4|B \text{ is even}) = \frac{1}{21}, P(A=5|B \text{ is even}) = \frac{2}{21}, P(A=6|B \text{ is even}) = \frac{2}{21}$$
$$P(A=7|B \text{ is even}) = \frac{4}{21}, P(A=8|B \text{ is even}) = \frac{3}{21}, P(A=9|B \text{ is even}) = \frac{2}{21}$$
$$P(A=10|B \text{ is even}) = \frac{2}{21}, P(A=11|B \text{ is even}) = \frac{2}{21}, P(A=12|B \text{ is even}) = \frac{1}{21}$$

Question 2:

Suppose that a fair coin is tossed independently n times. Determine the probability of obtaining exactly n-1 heads, given (a) that at least n-2 heads are obtained and (b) that heads are obtained on the first n-2 tosses.

solution:

Consider three events:

A: obtain exactly n-1 heads.

B: obtain at least n-2 heads.

C: heads are obtained on the first n-2 tosses.

(a)

$$P(A) = \frac{\binom{n}{n-1}}{2^n}$$
$$P(B) = \frac{\binom{n}{n}\binom{n}{n-1}\binom{n}{n-2}}{2^n}$$

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} = \frac{P(A)}{P(B)} = \frac{2n}{n^2+n+2}$$

(b)

Given the first n-2 tosses are all heads, then we need only 1 head in the rest of tosses.

Therefore,

$$P(A|C) = \frac{2}{2 \times 2} = \frac{1}{2}$$

Question 3:

Three palyers A,B,C take turns tossing a fair coin. Suppose that A tosses first, B tosses second and C tosses third. This cycle is repeated indefinitely until someone wins by being the first player to obtain a head. Determine the probability that each of three players will win.

solution:

If all loss in one cycle, the game will continue, and the probability is $(\frac{1}{2})^3 = \frac{1}{8}$.

$$P(A \text{ wins}) = \sum_{n=0}^{\infty} (\frac{1}{8})^n \times \frac{1}{2} = \frac{4}{7}$$

$$P(B \text{ wins}) = \sum_{n=0}^{\infty} (\frac{1}{8})^n \times \frac{1}{2} \times \frac{1}{2} = \frac{2}{7}$$

$$P(C \text{ wins}) = 1 - P(A \text{ wins}) - P(B \text{ wins}) = \frac{1}{7}$$