Probabilities

Question 1:

X and Y are two random variables and their joint probability mass function is shown in the following table, a is a constant value.

	x=1	x=2	x=3
y=1	a	2a	3a
y=2	0	4a	a
y=3	6a	a	2a

(a) Find the value of a.

(b) Find $p_Y(3)$.

(c) Consider a random variable $Z = XY^2$. Find $E[Z|Y = 2]$.

(d) Given $Y \neq 3$, are X and Y independent?

(e) Find the conditional variance of Y given that $X = 2$.

Solution:

(a)

$$\begin{aligned} 1 &= \sum_{x=1}^{3}\sum_{y=1}^{3} \\ &= a + 2a + 3a + 0 + 4a + a + 6a + a + 2a \\ &= 20a \end{aligned}$$

$\Rightarrow a = \frac{1}{20}$

(b)

$p_Y(3) = \sum_{x=1}^{3} p_{X,Y}(x, 3) = 6a + a + 2a = 9a = \frac{9}{20}$

(c)

$E[Z|Y = 2] = E[XY^2|Y = 2] = E[4X|Y = 2] = 4E[X|Y = 2])$

$p_{X|Y}(x|2) = \begin{cases} 0 & x = 1 \\ \frac{4a}{5a} = \frac{4}{5} & x = 2 \\ \frac{a}{5a} = \frac{1}{5} & x = 3 \end{cases}$

$E[X|Y = 2] = \frac{4}{5} \cdot 2 + \frac{1}{5} \cdot 3 = \frac{11}{5}$

$E[Z|Y = 2] = 4E[X|Y = 2] = 4 \cdot \frac{11}{5} = \frac{44}{5}$

(d)

$$p(Y = y|Y \neq 2) = \begin{cases} \frac{6}{11} & y = 1 \\ \frac{5}{11} & y = 2 \end{cases}$$

$$p(Y = y|X = 1, Y \neq 2) = \begin{cases} 1 & y = 1 \\ 0 & y = 2 \end{cases}$$

$$p(Y = y|X = 2, Y \neq 2) = \begin{cases} \frac{2}{3} & y = 1 \\ \frac{1}{3} & y = 2 \end{cases}$$

$$p(Y = y|X = 2, Y \neq 2) = \begin{cases} \frac{3}{4} & y = 1 \\ \frac{1}{4} & y = 2 \end{cases}$$

Since given $Y \neq 3$, the distribution of X is not the same given $X = x$, X and Y are not independent.

(e)

$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2, y) = 2a + 4a + a = 7a = \frac{7}{20}$

$$p_{Y|X}(y|2) = \frac{p_{X,Y}(2,y)}{p_X(2)} = \begin{cases} \frac{2/20}{7/20} = \frac{2}{7} & y = 1 \\ \frac{4/20}{7/20} = \frac{4}{7} & y = 2 \\ \frac{1/20}{7/20} = \frac{1}{7} & y = 3 \end{cases}$$

$E[Y|X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y|2) = 1 \cdot \frac{2}{7} + 2 \cdot \frac{4}{7} + 3 \cdot \frac{1}{7} = \frac{13}{7}$

$E[Y^2|X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y|2) = 1^2 \cdot \frac{2}{7} + 2^2 \cdot \frac{4}{7} + 3^2 \cdot \frac{1}{7} = \frac{27}{7}$

$var[Y|X = 2] = E[Y^2|X = 2] - (E[Y^2|X = 2])^2 = \frac{27}{7} - (\frac{13}{7})^2 = \frac{20}{49}$

Question 2:

Let X and Y be Gaussian random variables, with $X \sim \mathcal{N}(2,4)$ and $Y \sim \mathcal{N}(1,3)$.

(a) Find $P(X \leq 1.5)$ and $P(X > 2)$.

(b) Find the distribution of $\frac{Y-1}{2}$.

(c) Find $P(2 < X \leq \pi)$ and $P(X \leq Y)$.

Solution:

(a)

$$P(X \leq 1.5) = P(\tfrac{X-2}{2} \leq \tfrac{1.5-2}{2}) = \Phi(-0.25) = 1 - \Phi(0.25) = 1 - 0.5987 = 0.4013$$
$$P(X > 2) = 1 - P(X \leq 2) = 1 - P(\tfrac{X-2}{2} \leq \tfrac{2-2}{2}) = 1 - \Phi(0) = 1 - 0.5 = 0.5$$

(b)

$$E[\tfrac{Y-1}{2}] = \tfrac{1}{2}(E[Y] - 1) = 0$$
$$\text{var}[\tfrac{Y-1}{2}] = \tfrac{1}{4}\ \text{var}[Y] = 0.75$$
$$\Rightarrow \tfrac{Y-1}{2} \sim \mathcal{N}(0, 0.75)$$

(c)

$$P(2 < X < \pi) = P(\tfrac{2-2}{2} \leq \tfrac{X-2}{2} \leq \tfrac{\pi-2}{2}) = \Phi(0.5708) - \Phi(0) = 0.7159 - 0.5 = 0.2159$$
$$E[X-Y] = E[X] - E[Y] = 2 - 1 = 1$$
$$var[X-Y] = var[X] + var[Y] = 4 + 3 = 7$$

Thus, $X - Y \sim \mathcal{N}(1,7)$.

$$P(X \leq Y) = P(X - Y \leq 0) = P(\tfrac{(X-Y)-1}{\sqrt{7}} \leq \tfrac{0-1}{\sqrt{7}})$$
$$= \Phi(-0.378) = 1 - \Phi(0.378) = 1 - 0.6445 = 0.3555$$

Question 3:

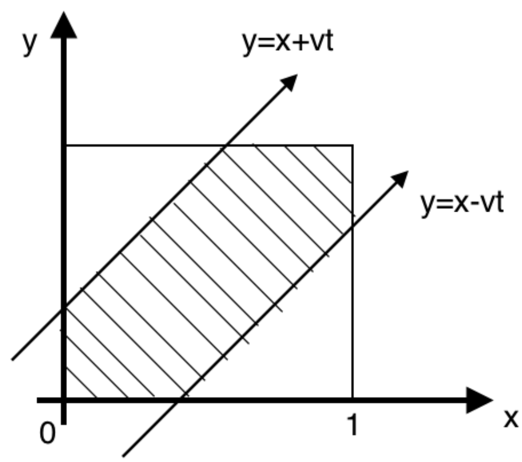
San Zhang drives a taxi back and forth at a constant speed v, along a road of length l. Assume that the location of the taxi at any time is uniformly distributed over the interval $(0, l)$. And the passenger occurs at a point uniformly distributed on the road. Supposing the location of the passenger and the location of the taxi are independent, and the U-turns is trival in time. Find the CDF and PDF of the taxis travel time T to the location of the passenger.

Solution:

Consider X and Y are the random variables, which are the location of the taxi and the passenger. Since X and Y are independent and uniformlt distributed over $[0, l]$,

$$fX, Y(x, y) = \begin{cases} \frac{1}{l^2} & 0 \le x, y \le l \\ 0 & \text{otherwise} \end{cases}$$

$P(T \le t) = P(\frac{|X-Y|}{v} \le t) = P(|X - Y| \le vt)$
$= P(-vt \le Y - X \le vt) = P(X - vt \le Y \le X + vt)$
$= \int_x \int_y fX, Y(x, y)dxdy, x - vt \le y \le x + vt$



CDF: $F_T(t) = \frac{1}{l^2} \times$ (shared area) $= \begin{cases} 0 & t < 0 \\ \frac{2vt}{l} - \frac{v^2t^2}{l^2} & 0 \le t \le \frac{l}{v} \\ 1 & t > \frac{l}{v} \end{cases}$

We can get PDF of T by differentiating the CDF:

$$f_T(t) = \begin{cases} \frac{2v}{l} - \frac{2v^2t}{l^2} & 0 \le t < \frac{l}{v} \\ 0 & \text{otherwise} \end{cases}$$

Question 4:

Consider a random variable X which is uniformly distributed between 0 and 1. On any given day, X denotes the probability that the weather is sunny with the probability p, and the status of the weather on different days is independent.

(a) Find the probability that it is sunny on a particular day. (Please notice the difference between "particular" and "given")

(b) We know that for the last n days, m days were sunny. Find the conditional PDF of X.

hint: $\int_0^1 p^k(1-p)^{n-k}dp = \frac{k!(n-k)!}{(n+1)!}$

Solution:

(a)

Consider A be the event that it is sunny on a particular day.

(b)

Consider B be the event that m days were sunny out of last n days.

We know that B is binomial, thus, $P(B|X=q) = \int_0^1 \binom{n}{m} q^m(1-q)^{n-m}$

$$\begin{aligned}
P(B) &= \int_0^1 P(B|X=p)fX(p)dp \\
&= \int_0^1 \binom{n}{m} p^m(1-p)^{n-m} f_X(p)dp \\
&= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
\end{aligned}$$

$$\begin{aligned}
f_{X|B}(p) &= \frac{P(B|X=p)fQ(p)}{P(B)} \\
&= \frac{p^m(1-p)n-m}{\frac{m!(n-m)!}{(n+1)!}}, 0 \leq p \leq 1, m \leq n
\end{aligned}$$

Question 5:

(a) First roll a fair six-sided die, and then flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of Heads.

(b) What if you roll two dice instead of one, and then flip a fair coin the number of times shown by the sum of two dies?

Solution:

(a)

Suppose X_i be the random variables, which is the result of flipping a coin at the ith flip.

$$X_i = \begin{cases} 1 & head \\ 0 & tail \end{cases}$$

X_i are independent Bernoulli random variables with $p = \frac{1}{2}$, in which N is the number of coin flips.

$$E[X_i] = \frac{1}{2}, var[X_i] = \frac{1}{4}.$$

W know that N is discrete uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

$$E[N] = \frac{1+6}{2} = \frac{7}{2}, var[N] = \frac{(6-1+1)^2 - 1}{12} = \frac{35}{12}.$$

Suppose H is the number of heads. When N = n, the conditional distribution of H is binomial with parameters $n = n$ and $p = \frac{1}{2}$.

$$E[H|N = n] = \frac{n}{2}, var[H|N = n] = \frac{n}{4}.$$

Replace n with N, we have

$$E[H|N] = \frac{N}{2}, var[H|N] = \frac{N}{4}.$$

$$E[H] = E[E[H|N]] = E[\frac{N}{2}] = \frac{E[N]}{2} = \frac{7}{4}$$

$$var(H) = E[var(H|N)] + var(E[H|N]) = E[\frac{N}{4}] + var[\frac{N}{2}] = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot \frac{35}{12} = \frac{77}{48}$$

(b)

This experience is just two independent process of in **a**. Thus,

$$E[H'] = 2E[H] = \frac{7}{2}, var[H'] = 2var[H] = \frac{77}{24}$$

.

Question 6:

Given a Poisson process with parameter λ, and an exponential independentlly random variable T with parameter v. Consider a time interval $[0, T]$, find the PMF of the number of Poisson arrivals.

Solution:

Consider Poisson arrivals as event A, $A \sim \text{Poisson}(\lambda)$, $p(n|T = t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

Since A is exponential variable, $f_T(t) = ve^{-vt}, v \geq 0$.

$$\begin{aligned} P(N = n) &= \int_0^T P(N = n|T = t) f_T(t) dt \\ &= \int_0^T \frac{(\lambda t)^n e^{-\lambda t}}{n!} \times ve^{-vt} dt \\ &= \frac{(\lambda^n v)}{n!} \int_0^T t^n e^{-(\lambda + v)t} dt \end{aligned}$$

Question 7:

Tom and John alternatively play a game (Tom starts first). Assume that the score obtained at different times are independent and scores of 2 is a loss. The scores will not be accumulated in the following games. At any time, the score obtained of whoever is playing is a random variable S with the following PMF:

$$p_S(s) = \begin{cases} \frac{1}{3} & \text{for } s = -1 \quad (1) \\ \frac{1}{2} & \text{for } s = 1 \quad (2) \\ \frac{1}{6} & \text{for } s = 2 \quad (3) \\ 0 & \text{otherwise} \quad (4) \end{cases}$$

(a) A round consists of two plays, first Tom then John. They keep playing until the first time when Tom has a loss and then immediately John has a loss. Find the PMF of the total number of rounds played.

(b) Z is the time at which John has his fifth loss. Find the PMF for Z.

(c) N is the number of rounds when Tom and John both have won at least once. Find the expectation of N.

Solution:

(a)

We know from the PMF of S that, the probability of loss is $\frac{1}{6}$.

The event that Tom has a loss and then immediately John has a loss can only happened in one round where both of them lose, in which the probability $p = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

Supposing X is the radom variable of the total number of rounds played.

$$p_X(X) = (1-p)^{x-1}p = \frac{35^{x-1}}{36^x}, x = 1, 2, \ldots$$

(b)

Supposing T and J are two random variables of the number of games Tom and John respectively played until John has his fifth loss.

$$p_J(j) = \binom{j-1}{5-1}(\frac{1}{6})^5(1-\frac{1}{6})^{j-5}, j = 5, 6, 7, \ldots$$

We don't need to care the result of Tom played, and know that Z = 2J.

$$p_Z(z) = \binom{\frac{z}{2}-1}{5-1}(\frac{1}{6})^5(1-\frac{1}{6})^{\frac{z}{2}-5}, z = 10, 12, 14, \ldots$$

(c)

Consider the following events:

A_1 : both lose at the first round

A_2 : both win at the first round

A_3 : only one of them wins at the first round

We have $P[A_1] = (\frac{1}{6})^2 = \frac{1}{36}$

$P[A_2] = (1 - \frac{1}{6})^2 = \frac{25}{36}$

$P[A_1] = \frac{1}{6}(1 - \frac{1}{6}) = \frac{5}{36}$

If A_1, we need to start from the scrath at time 2, thus, $E[N|A_1] = 1 + E[N]$.

If A_2, we arrive at where we what, thus, $E[N|A_2] = 1$.

If A_3, we expect the time that another person wins, which is geometricly distributed with the mean of $\frac{1}{\frac{5}{6}}$. Therefore, $E[N|A_2] = 1 + \frac{6}{5} = \frac{11}{5}$.

$$\begin{aligned} E[N] &= E[N|A_1]P(A_1) + E[N|A_2]P(A_2) + 2E[N|A_3]P(A_3) \\ &= (1 + E[N]) \cdot \frac{1}{36} + 1 \cdot \frac{25}{36} + 2\frac{11}{5} \cdot \frac{5}{36} \\ &= \frac{1}{36}E[N] + \frac{4}{3} \end{aligned}$$

$\Rightarrow E[N] = \frac{48}{35}$

Question 8:

Some of the errors made by machines can be repaired by the workers. Suppose that there are N errors caused by machines work, and we model N as a Poisson random variable with expectation λ. Suppose that each error is repaired with probability p independently of the repair of other errors. Let K denote the number of errors that are repaired.

(a) Suppose $k \leq n$, find $P[K = k|N = n]$.

(b) For any given k, $k \geq 0$, what is the (unconditional) probability that exactly $K = k$ errors are repaired?

Simplifying your result with the hints:

$$\lambda^n = \lambda^k \cdot \lambda^{n-k}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

(c) What is the expected number of repaired errors?

Solution:

(a)

Given that there are n errors, the number K of repairs is binomially distributed with parameters n and p. It follows that

$$P[K = k|N = n] = \tbinom{n}{k} p^k (1-p)^{n-k}$$

(b)

The number of errors cannot be smaller than the number of repairs, thus, $n \geq k$.

Write $q = 1 - p$, we have,

$$\begin{aligned}
P[K = k] &= \sum_{n=k}^{\infty} P[K = k|N = n]P[N = n] \\
&= \sum_{n=k}^{\infty} \tbinom{n}{k} p^k q^{n-k} \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\
&= e^{-\lambda} \frac{(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda q)^{n-k}}{(n-k)!} \\
&= e^{-\lambda} \frac{(\lambda p)^k}{k!} e^{\lambda q} \\
&= e^{-\lambda(1-q)} \frac{(\lambda p)^k}{k!} \\
&= e^{-\lambda p} \frac{(\lambda p)^k}{k!}
\end{aligned}$$

(c)

The distribution of the number of repairs is Poisson with parameter λp, thus, $E(K) = \lambda p$