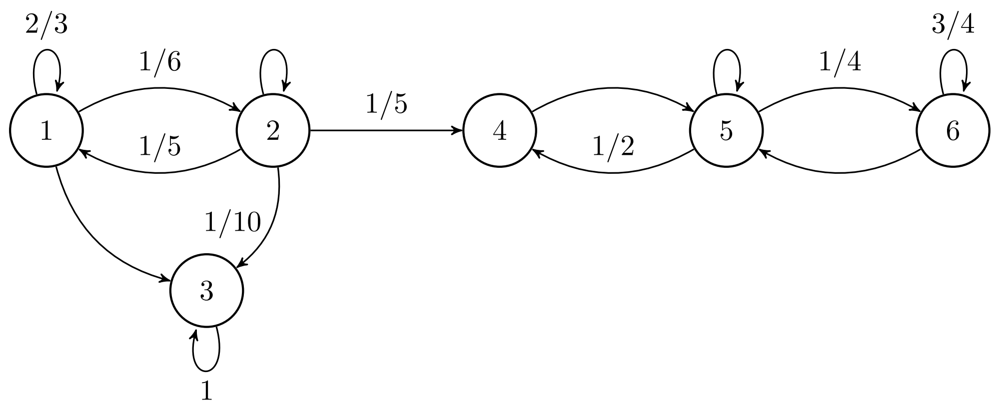
Markov Chain

Question 1:

Given the Markov chain model:



(a) Complete the missing probabilities in the above chain.

(b) For each state 1-6, identify whether it is recurrent or transient.

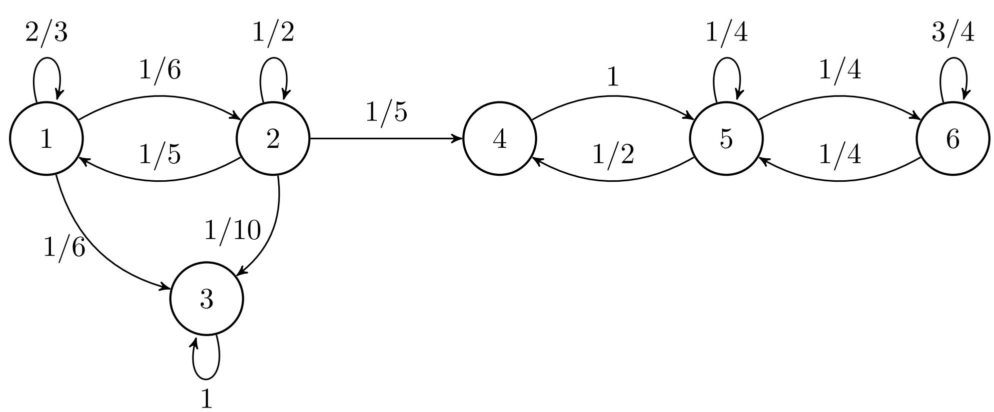
(c) If the state is 1 at Monday, what is the probability that it will be state 5 at Thursday the same week?

(d) If the state is 4 today, will the chain be converge to steady state? If so, give the probability. If not, give the reason.

(e) If the state is 1 today, give the probability that state 6 will never arrive in the future?

Solutions:

(a)



(b)

Recurrent states: 1,2.

Transient states: 3,4,5,6.

(c)

There are only one way from state 1 to state 5 in three steps.

$$\begin{aligned} P(S_4 = 5|S_1 = 1) &= P(S_2 = 2|S_1 = 1) \times P(S_3 = 4|S_2 = 2) \times P(S_4 = 5|S_3 = 4) \\ &= \frac{1}{6} \times \frac{1}{5} \times 1 \\ &= \frac{1}{30} \end{aligned}$$

(d) It will stay in recurrent class $4, 5, 6$.

$$\begin{cases} \pi_4 \times 1 = \pi_5 \times \frac{1}{2} \\ \pi_6 \times \frac{1}{4} = \pi_5 \times \frac{1}{4} \\ \pi_1 = 0 \\ \pi_2 = 0 \\ \pi_3 = 0 \\ \sum_{i=1}^{6} \pi_i = 1 \end{cases}$$

$\Rightarrow \pi_1 = 0, \pi_2 = 0, \pi_3 = 0, \pi_4 = \frac{1}{5}, \pi_5 = \frac{2}{5}, \pi_6 = \frac{2}{5}$.

(e) Let One and Two be the events where state 6 will never arrive when starting from state 1 or state 2 respectively. Thus, it will ends in state 3.

$$\begin{cases} P(One) = \frac{2}{3} \times P(One) + \frac{1}{6} \times P(Two) + \frac{1}{6} \times 1 \\ P(Two) = \frac{1}{5} \times P(One) + \frac{1}{2} \times P(Two) + \frac{1}{10} \times 1 \end{cases}$$

$\Rightarrow P(One) = \frac{3}{4}$.

Question 2:

Consider $X_1, X_2, \ldots, X_{20}$ independent random variables, which are uniformly distributed over the interval $[0, 1]$. Estimate $P(X_1 + \cdots + X_{20} \geq 7)$ using Markov inequality, Chebyshev inequality and CLT respectively.

Solutions:

Let $X = \sum_{i=1}^{20} X_i$, then $E[X] = 20E[X_i] = 10, Var[X] = 20Var[X_i] = 20 \cdot \frac{1}{12} = \frac{5}{3}$.

(Markov Inequality)

$P(X \geq 7) \leq \frac{E[Z]}{7} = \frac{10}{7}$

Acutally, $P(X \geq 7) \leq 1$.

(Chebyshev Inequality)

$$\begin{aligned}
&P(|X-\mu| \geq K\delta) \leq \frac{1}{K^2} \\
\Rightarrow &P(X \geq \mu + K\delta \mid\mid X \leq \mu - K\delta) \leq \frac{1}{K^2} \\
\Rightarrow &P(X \leq \mu - K\delta) \leq \frac{1}{2K^2} \\
\Rightarrow &P(X \geq \mu - K\delta) \geq 1 - \frac{1}{2K^2} \\
\Rightarrow &P(X \geq 10 - K\sqrt{\frac{5}{3}}) \geq 1 - \frac{1}{2K^2}
\end{aligned}$$

Thus, $10 - K\sqrt{\frac{5}{3}} = 7 \Rightarrow K^2 = \frac{27}{5}$,

$\Rightarrow P(X \geq 7) \geq \frac{49}{54}$

(CLT)

$P(X \geq 7) = P(Y \geq \frac{7-\mu}{\delta}) = P(X \geq -2.32) = \Phi 2.32 = 0.9898$